A refined Born approximation

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ABSTRACT

The proposed method of computing scattering amplitudes and cross-sections is by far more efficient but not much more cumbersome than the Born approximation. It applies to the cases of medium strong couplings in high as well as low energy region.

1. A REFINED BORN APPROXIMATION FOR PARTIAL WAVES

a. Asymptotic fit

Let us explain first the Refined Born Approximation (RBA) method on an example of the non-relativistic scattering theory using a partial wave expansion. The well known Lippmann-Schwinger equation for partial waves is

$$\psi_{lp}(r) = \hat{j}_l(pr) + \int_0^\infty dr' G_{lp}(r,r') U(r') \psi_{lp}(r') \quad (1)$$

where a standard notation of Taylor's [2] book has been used. The usual Born approximation consists in solving this integral equation for the partial waves ψ_{lp} by iteration inserting, as the zero-order approximation, into the integrand to the right of the Riccati-Bessel functions

$$\psi_{lp}^{(0)}(r) = \hat{j}_l(pr) \quad (2)$$

which yields for the left hand side $\psi_{lp}^{(1)}$, inserting it again into the integrand to obtain $\psi_{lp}^{(2)}$, and so forth. To each order of approximation there corresponds the approximative scattering amplitude

$$f_l^{(n)}(p) = \frac{-1}{p^2} \int_0^\infty dr\, \hat{j}_l(pr)\, U(r)\, \psi_{lp}^{(n)}(r) \quad (3)$$

(see also Taylor [2]).

The simplest version of our RBA method (proposed for the first time in 1974, see [1]) consists in the following : instead of the input (2) we assume slightly more generally

$$\psi_{lp}^{(0)} = (1+\alpha)\, \hat{j}_l(pr) \quad (4)$$

where α is a parameter to be fitted later by a kind of a bootstrap requirement. In the case of a weak interaction α is small but the method will be shown to be applicable also for medium strong interactions.

In consequence of (4) the scattering amplitudes, in all orders of approximation $f_l^{(n)}(p)$, become functions of this parameter which may be fixed by the requirement

$$f_l^{(n)}(p) = f_l^{(n-1)}(p) \quad (5)$$

for a certain order n. This is a self-consistency fit because if $\psi_{lp}^{(0)}$ for a certain value of α were the correct solution, then (5) would be satisfied automatically and simultaneously in all orders of approximation. Inasmuch as (4) is not an exact solution, we may at least try to make it as similar as possible to a true solution in the desired order of approximation. This method is no more equivalent to a power series expansion in the coupling parameter (hidden in the potential U). On an example of a square well it will be shown that even in the second approximation

$$f_l^{(2)}(p) = f_l^{(1)}(p) \quad (5')$$

it yields satisfactory results even in the case of a comparatively strong coupling case (coupling constant > 1) and for low p where the ordinary (second) Born approximation fails completely.

b. A fit at the origin

Sometimes even the computation of a second order scattering amplitude is very cumbersome so that a parameter fit from (5') presents considerable difficulties. In such cases it would be desirable to possess a possibility of a self-consistent fit of a parameter without the necessity of performing more than one iteration. This is possible and particularly simple by fitting the parameter α appearing in (4) from the requirement that the partial waves of the first and the zeroth order coincide at the origin ($r \to 0$). More precisely, we may equate the coefficients of the lowest non-vanishing terms of the power series expansions

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$$\psi_{lp}^{(0)}(r) = \sum_{n=0} c_n^{(0)} r^{n+l+1},$$

$$\psi_{lp}^{(1)}(r) = \sum_{n=0} c_n^{(1)} r^{n+l+1} \qquad (6)$$

i.e. by equating

$$C_0^{(0)} = C_0^{(1)} \qquad (7)$$

This is again a self-consistent fit since if the zeroth order partial wave were guessed correctly to be a true solution then the first iterated function would be identical with it everywhere; also at the origin. With the condition (7) the first order iterated function imitates the true solution as much as possible at least close to the origin. This is sufficient if the potential is shortranged.
Introducing the value of α from (7) into the first order partial wave one is able to compute the first order scattering amplitude from (3).

c. Many-parameter fits

If the one-parameter fit in the first order of approximation may be insufficient to obtain the desired accuracy, we may improve the result without going over to higher order amplitudes by introducing more parameters.
E. g. we may introduce not only one single $\hat{j}_{lp}$ for the partial wave $\psi_{lp}^{(0)}$ but a linear combination of a few adjacent waves

$$\psi_{lp}^{(0)} = \sum_{\lambda=l-k}^{l+k} (\delta_{l\lambda} + \alpha_{l\lambda}) \hat{j}_{\lambda p} \qquad (8)$$

Performing an iteration we may fit the $2k+1$ parameters $\alpha_{l\lambda}$ from the requirements

$$C_n^{(0)} = C_n^{(1)} \quad \text{for} \quad n = 0, 1, \ldots 2k \qquad (9)$$

where again C_n are the coefficients of the power series expansions (6).

2. A REFINED BORN APPROXIMATION WITHOUT PARTIAL WAVE EXPANSION

Sometimes it may be desirable to have a scattering amplitude in a closed form and not necessarily analysed into partial waves. For instance, we may be particularly interested into a differential cross-section under a given angle θ, e.g. forward or backward scattering. Some possibilities of self-consistent parameter fits exist also in this case.
Let us split the Schrödinger function into an incident plane wave and a scattered wave

$$\psi(\vec{r}) = e^{i\vec{p}\vec{r}} + \gamma(\vec{r}) \qquad (10)$$

and replace the Schrödinger equation by an integral equation describing scattering processes. This last may be written as an equation for the scattered part γ

$$\gamma(\vec{r}) = -\int d^3r'\, G^{ret}(|\vec{r} - \vec{r}'|)\, U(\vec{r}')\, [e^{i\vec{p}\vec{r}'} + \gamma(\vec{r}')] \qquad (11)$$

where the Green function is

$$G^{ret}(r) = \frac{1}{4\pi} \frac{e^{ipr}}{r} = -\frac{1}{(2\pi)^3} \int d^3k \frac{e^{i\vec{k}\vec{r}}}{p^2 - k^2 + i\epsilon} \qquad (12)$$

The usual Born method consists in assuming as a zero-order approximation the ingoing plane wave, i.e. in putting

$$\psi^{(0)} = e^{i\vec{p}\vec{r}} \quad \text{or} \quad \gamma^{(0)} = 0 \qquad (13)$$

whereas we may generalize it slightly by putting

$$\psi^{(0)} = (1+\alpha)\, e^{i\vec{p}\vec{r}} \quad \text{or} \quad \gamma^{(0)} = \alpha e^{i\vec{p}\vec{r}} \qquad (14)$$

compute the first and possibly also the higher order corrections $\gamma^{(1)}, \gamma^{(2)}, \ldots$ from (11) and fit the parameter at the origin or at infinity. The fit at the origin ($r \to 0$) may be obtained from the requirement

$$\gamma^{(n)}(0) = \gamma^{(n-1)}(0) \qquad (15)$$

Alternatively, if we are interested in a particular differential cross-section, say cross-section for scattering at a given angle θ, we may compute first the corresponding scattering amplitude $f^{(n)}_{(p,\theta)}$ using the asymptotic relation

$$\gamma^{(n)}(\vec{r}) \to \frac{e^{ipr}}{r} f^{(n)}(p, \theta) \qquad (16)$$

and fit the parameter α from the requirement

$$f^{(n)}(p, \theta) = f^{(n-1)}(p, \theta) \qquad (17)$$

with the given, *fixed* value of θ. In this way we shall obtain, for every angle θ, a different result $\alpha_{p,\theta}$ but this does not contradict the assumption α to be originally a constant parameter and not a function of θ because it is not meant to be an exact but only an approximate solution. Of course, one might then improve our ansatz (14) by introducing a variable α, obtained by interpolation between a few values of α derived primarily for some angles, say $\alpha = 0, \pi/2, \pi$.
The fit at the origin (15) may be performed in the first approximation

$$\gamma^{(1)}(0) = \gamma^{(0)}(0) = \alpha \qquad (15')$$

and is very simple although the agreement with the exact solution may be very poor in this case. It is possible to improve the agreement by introducing some more parameters, e.g.

$$\gamma^{(0)} = (\alpha + \beta \frac{r}{r_0}) e^{i\vec{p}\vec{r}} \qquad (18)$$

and fit the parameters α and β from the requirements

$$\gamma^{(2)}(0) = \gamma^{(1)}(0) = \gamma^{(0)}(0) \qquad (19)$$

On the other hand, the asymptotic fits at infinity (17) require the computation of at least the second order scattering amplitude $f^{(2)}(p, \theta)$ which is comparatively easy for the case of a forward (backward) scattering but may be cumbersome in general.

3. AN EXAMPLE : A SQUARE WELL POTENTIAL

Let us consider a very simple example of a scattering of extremely slow particles ($p \to o$) on a square well potential

$$V = \begin{cases} g/r_0 & \text{for } r_0 > r \\ 0 & \text{for } r \geqslant r_0 \end{cases} \tag{20}$$

Introducing the notation

$$G = -Ur_0^2 = -2mVr_0^2 \tag{21}$$

the equation (11) simplifies considerably in the limit $p \to 0$

$$\chi(\vec{r}) = \frac{G}{4\pi r_0^2} \int d^3 r' \frac{1}{|\vec{r} - \vec{r}'|} \{1 + \chi(\vec{r}')\} \tag{22}$$

The exact solution of this equation and, consequently, the s-wave scattering length, equal minus amplitude at zero momentum, is exactly known (see Taylor [2])

$$a_0 = -f_0(0) = 1 - \frac{\operatorname{tg} G^{1/2}}{G^{1/2}} \tag{23}$$

so that the reliability of the RBA method may be checked.

For $p \to 0$ the ansatz (14) becomes simply

$$\psi^{(0)} = 1 + \alpha \qquad \text{or} \qquad \gamma^{(0)} = \alpha \tag{14'}$$

Introducing it into the right hand side of (22) we get the result

$$\gamma^{(1)}(\vec{r}) = \begin{cases} \frac{G}{2}(1+\alpha)\left[1 - \frac{1}{3}\left(\frac{r}{r_0}\right)^2\right] & \text{for } r < r_0 \\ \frac{G}{3}\frac{r_0}{r}(1+\alpha) & r \geqslant r_0 \end{cases} \tag{24}$$

The asymptotic form for $r \gg r_0$ yields the following expression for the scattering amplitude

$$f^{(1)} = \frac{1}{3} G (1+\alpha) r_0 \tag{25}$$

Introducing (24) again into the right hand side of (22) we find the following asymptotic forms : for $r \to 0$

$$\gamma^{(2)}(0) = \frac{G}{2}\left[1 + \frac{5}{12}(1+\alpha) G\right] \tag{26}$$

and for $r \gg r_0$

$$f^{(2)} = -\frac{G}{3}\left[1 + \frac{2}{5}(1+\alpha) G\right] r_0 \tag{27}$$

The simple fit (15') yields

$$\alpha = \frac{G}{2 - G} \tag{28}$$

whence the amplitude (25) is seen to possess a pole for $G = 2$. It is seen that this simple ansatz reproduces already the first pole of the exact solution (23) with an accuracy of about 10 %. Indeed, we get a pole for $G^{1/2} = 1.41$ instead of $G^{1/2} = \pi/2 = 1.57$ (see fig. 1).

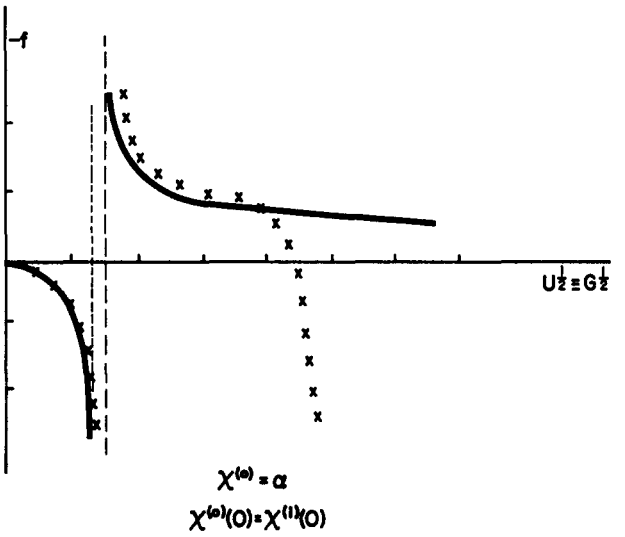

Fig. 1.

By fitting the parameter α from the more elaborate condition

$$\gamma^{(2)}(0) = \gamma^{(1)}(0) \tag{29}$$

we get from (24) and (26)

$$\alpha = \frac{G}{\frac{12}{5} - G} \tag{30}$$

a result very similar to (28). By introducing this value into (27) we obtain

$$f^{(2)} = \frac{G}{3} \frac{1 - \frac{1}{60} G}{1 - \frac{5}{12} G} r_0 \tag{31}$$

In this case one gets a pole at $G^{1/2} = 1.55$ so that the error in comparison with $\pi/2$ is only about 1.3 %. Moreover, there appears also a zero of the amplitude (besides that $G = 0$) which, however, is situated far away from the true position from (23) (see fig. 2).

A still better agreement of the position of the first pole is obtained by fitting the parameter from the asymptotic condition

$$f^{(2)} = f^{(1)} \tag{17'}$$

namely

$$\alpha = \frac{G}{\frac{5}{2} - G} \tag{32}$$

whence $G^{1/2} = 1.58$ differing from $\pi/2$ only about 7 ‰.

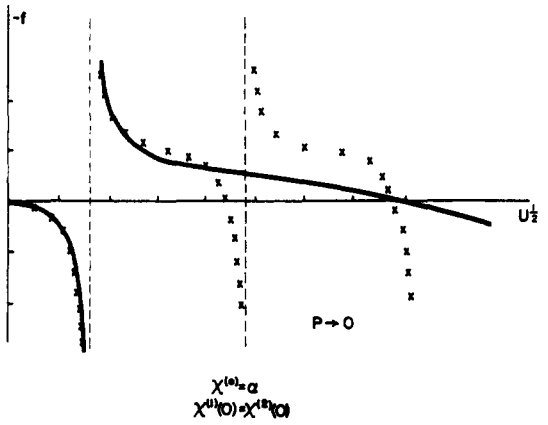

Fig. 2.

On the other hand, the corresponding amplitude

$$f^{(2)} = \frac{G}{3} \frac{1}{1 - \frac{2}{5}G} r_0 \qquad (33)$$

does not exhibit any zero except for $G = 0$.

The accuracy of the approximation may be still improved considerably by introducing a two-parameter fit (18), (19). In this case one gets

$$f^{(2)} = -\frac{1}{240} \frac{G^2 - 240G + 4800}{G^2 - 27G + 60} G r_0 \qquad (34)$$

exhibiting two poles and zeros (besides $G = 0$). The zeros are at the values 0, 4.69, 14.8 whereas the poles appear at $G^{1/2} = 1.562$ and $G^{1/2} = 4.96$ exhibiting errors of about 5 ‰ and 5 % only in comparison with the first two poles of the exact solution (23). Thus, the results are very satisfactory up to the value of $G \approx 35$ (see fig. 3).

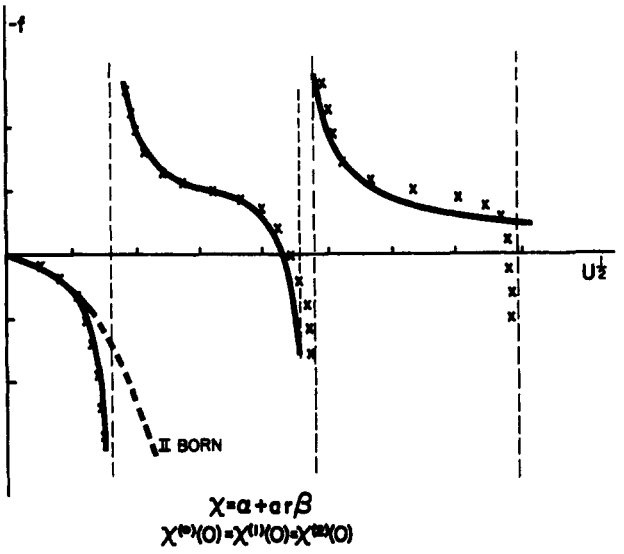

Fig. 3.

The square well potential may be considered as a first, very crude model of nuclei. Assuming the radius of the well r_0 to be 2 fermi and the depth 50 MeV is equivalent to assuming the value

$$G = 13.4 \quad \text{or} \quad G^{1/2} = 3.6 \qquad (35)$$

showing that our method of approximations applies well to the strong coupling case for light nuclei.

Let us point out that the case $p \to 0$) is particularly unfavourable as regards the reliability of the ordinary Born approximations. Moreover, although the RBA method takes advantage of the iteration procedure, it has nothing to do with the power series expansion in terms of the coupling constant $G^{1/2}$. The appearance of rational fractions like that in (34) reminds rather of the Padé approximations.

The above method may be applied also for relativistic equations and may be generalized for the case of the quantum field theory.

REFERENCES

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